Mathematical Induction 3

1. Prove that : $\frac{d^n}{dx^n}\sin x = sin\left(x+\frac{n\pi}{2}\right)$

Let $P(n)$ be the proposition: $\frac{d^n}{dx^n}\sin x = sin\left(x+\frac{n\pi}{2}\right)$.

We like to use the Principle of Mathematical Induction to prove that $P(n)$ is true $\forall n\epsilon$.

For $P(1)$, $\frac{d}{dx}\sin x = \cos x = sin\left(x+\frac{\pi}{2}\right)$. $\therefore P(1)$ is true.

Assume $P(k)$ is true for some $k\epsilon$, that is, $\frac{d^k}{dx^k}\sin x = sin\left(x+\frac{k\pi}{2}\right)$ ... (*)

For $P(k+1)$, $\frac{d^{k+1}}{dx^{k+1}}\sin x = \frac{d}{dx}\left[\frac{d^k}{dx^k}\sin x\right] = \frac{d}{dx}\left[sin\left(x+\frac{k\pi}{2}\right)\right]$, by (*)

$$= cos\left(x+\frac{k\pi}{2}\right) = sin\left[\frac{\pi}{2}+\left(x+\frac{k\pi}{2}\right)\right] = sin\left[x+\frac{(k+1)\pi}{2}\right]$$

$\therefore P(k+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true $\forall n\epsilon$.

2. Prove $1\times 3+3\times 5+5\times 7+\cdots+(2n-1)(2n+1)=\frac{n}{3}(4n^2+6n-1)$

by mathematical induction.

The magic is to change the right hand side of the proposition to:

$$P(n): 1\times 3+3\times 5+5\times 7+\cdots+(2n-1)(2n+1)=\frac{1}{6}[(2n-1)(2n+1)(2n+3)+3]$$

For $P(1)$, L.H.S. $=1\times 3 = 3$, R.H.S. $=\frac{1}{6}[1\times 3\times 5+3]=3$

Assume $P(k)$ is true for some $k\in\boldsymbol{N}$, that is

$$1\times 3+3\times 5+5\times 7+\cdots+(2k-1)(2k+1)=\frac{1}{6}[(2k-1)(2k+1)(2k+3)+3] \ldots (1)$$

For $P(k+1)$,

$1\times 3+3\times 5+5\times 7+\cdots+(2k-1)(2k+1)+(2k+1)(2k+3)$

$=\frac{1}{6}[(2k-1)(2k+1)(2k+3)+3]+(2k+1)(2k+3)$, by (1).

$=\frac{1}{6}[(2k-1)(2k+1)(2k+3)+6(2k+1)(2k+3)+3]$

$=\frac{1}{6}\{(2k+1)(2k+3)[(2k-1)+6]+3\}$

$=\frac{1}{6}\{(2k+1)(2k+3)(2k+5)+3\}$

$\therefore P(k+1)$ is true.

By the principle of mathematical induction, $P(n)$ is true for all $n \in \boldsymbol{N}$.
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Why: Let $u_i = (2n-1)(2n+1)$, $\quad v_i = (2n-1)(2n+1)(2n+3)$

It is not difficult to prove $\quad u_i = \frac{1}{6}[v_i - v_{i-1}]$.

Hence $\sum_{i=1}^{n} u_i = \frac{1}{6}\sum_{i=1}^{n}[v_i - v_{i-1}] = \frac{1}{6}(v_n - v_0) = \frac{1}{6}[(2n-1)(2n+1)(2n+3)+3]$

3. Prove $1^2 + 2^2 + 3^2 + \cdots + (2n)^2 = \frac{n(2n+1)(4n+1)}{3}$ by any method.

Method 1 Mathematical Induction

Let $P(n): 1^2 + 2^2 + 3^2 + \cdots + (2n)^2 = \frac{n(2n+1)(4n+1)}{3}$

For $P(1)$, $LHS = 1^2 + 2^2 = 5 = \frac{1(2+1)(4+1)}{3} = RHS$, $\quad \therefore P(1)$ is true.

Assume $P(k)$ is true for some $k \in \boldsymbol{N}$, that is $\quad 1^2 + 2^2 + 3^2 + \cdots + (2k)^2 = \frac{k(2k+1)(4k+1)}{3}$ ... (1)

For $P(k+1)$,

$$1^2 + 2^2 + 3^2 + \cdots + (2k)^2 + (2k+1)^2 + (2k+2)^2$$

$$= \frac{k(2k+1)(4k+1)}{3} + (2k+1)^2 + (2k+2)^2 \quad \text{, by (1)}$$

$$= \frac{k(2k+1)(4k+1) + 3(2k+1)^2 + 3[2k+2]^2}{3} = \frac{(2k+1)[k(4k+1)+3(2k+1)] + 12[k+1]^2}{3}$$

$$= \frac{(2k+1)[4k^2+7k+3] + 12[k+1]^2}{3} = \frac{(2k+1)(k+1)(4k+3) + 12[k+1]^2}{3} = \frac{(k+1)[(2k+1)(4k+3)+12(k+1)]}{3}$$

$$= \frac{(k+1)[8k^2+22k+15]}{3} = \frac{(k+1)(2k+3)(4k+5)}{3}$$

$\therefore P(k+1)$ is true.

By the principle of mathematical induction, $P(n)$ is true for all $n \in \boldsymbol{N}$.

Method 2 Difference method (1)

Let $v_{2k} = \frac{k(2k+1)(4k+1)}{3} = \frac{(2n)(2k+1)(2(2k)+1)}{6}, v_{2k-1} = \frac{(2k-1)(2k)(4k-1)}{6}$

$v_{2k} - v_{2k-1} = \frac{2k(2k+1)(4k+1) - (2k-1)(2k)(4k-1)}{6} = (2k)^2$

Taking summation $\sum_{2k=1}^{2n}[v_{2k} - v_{2k-1}] = \sum_{2k=1}^{2n}(2k)^2$

$\therefore \sum_{2k=1}^{2n}(2k)^2 = v_{2n} - v_0$

$\therefore 1^2 + 2^2 + 3^2 + \cdots + (2n)^2 = \frac{n(2n+1)(4n+1)}{3}$

Method 3 Difference method (2)

Consider $(k+1)^3-(k-1)^3=6k^2+2$

$\sum_{k=1}^{2n}[(k+1)^3-(k-1)^3]=6\sum_{k=1}^{2n}k^2+\sum_{k=1}^{2n}2$

$(2n+1)^3+(2n)^3-1^3-0^3=6\sum_{k=1}^{2n}k^2+4n$

$\therefore \sum_{k=1}^{2n}k^2=1^2+2^2+3^2+\cdots+(2n)^2=\frac{(2n+1)^3+(2n)^3-1^3-0^3-4n}{6}=\frac{8n^3+6n^2+n}{3}=\frac{n(2n+1)(4n+1)}{3}$

4. $\int_0^{2\pi} sin^{2n}xdx=\frac{2\pi(2n)!}{(n!)^2(4^n)}$

Let $P(n)$: $I_n=\int_0^{2\pi} sin^{2n}xdx=\frac{2\pi(2n)!}{(n!)^2(4^n)}$

For $P(0)$, $LHS=\int_0^{2\pi}dx=2\pi=RHS$, $\therefore P(0)$ is true.

Assume $P(k)$ is true for some $k\in\boldsymbol{N}$, that is $I_k=\int_0^{2\pi} sin^{2k}xdx=\frac{2\pi(2k)!}{(k!)^2(4^k)}\ldots(1)$

For $P(k+1)$,

$I_{k+1}=\int_0^{2\pi} sin^{2(k+1)}xdx=-\int_0^{2\pi} sin^{2k+1}x\, d(\cos x)$

$=-[sin^{2n+1}x(\cos x)]_0^{2\pi}+\int_0^{2\pi}\cos x\ d(sin^{2k+1}x)$ (Integration by parts)

$=0+(2k+1)\int_0^{2\pi}\cos^2 x\, sin^{2k}x\ dx(2k+1)\int_0^{2\pi}(1-\sin^2 x)sin^{2k}x\ dx$

$=(2k+1)\left[\int_0^{2\pi} sin^{2k}x\ dx-\int_0^{2\pi} sin^{2(k+1)}xdx\right]$

$=(2k+1)\,I_k-(2k+1)\,I_{k+1}$

Solve for I_{k+1} and by (1) we have

$I_{k+1}=\frac{(2k+1)}{(2k+2)}I_k=\frac{2\pi(2k)!}{(k!)^2(4^k)}\times\frac{(2k+1)}{(2k+2)}=\frac{2\pi(2k)!}{(k!)^2(4^k)}\times\frac{(2k+1)(2k+2)}{4(k+1)^2}=\frac{2\pi[2(k+1)]!}{[(k+1)!]^2(4^{k+1})}$

$\therefore P(k+1)$ is true.

By the Principle of mathematical induction, $P(n)$ is true for all $n\in\boldsymbol{N}\cup\{\boldsymbol{0}\}$.

5. Prove:

$(1^5+2^5+\cdots+n^5)+(1^7+2^7+\cdots+n^7)=2(1+2+\cdots+n)^4$.

Redefine the proposition into equivalent proposition:

Let $P(n)$: (a) $1+2+\cdots+n=\frac{n(n+1)}{2}$

(b) $(1^5+2^5+\cdots+n^5)+(1^7+2^7+\cdots+n^7)=\frac{1}{8}n^4(n+1)^4$

For $P(1)$: (a) LHS = 1= $\frac{1(1+1)}{2}$ = RHS

(b) LHS $=1^5+1^7=2=\frac{1}{8}1^4(1+1)^4$ = RHS

$\therefore P(1)$ is true.

Assume $P(k)$ is true for some $k \in \boldsymbol{N}$, that is

(a) $1+2+\cdots+k=\frac{k(k+1)}{2} \ldots(1)$

(b) $(1^5+2^5+\cdots+k^5)+(1^7+2^7+\cdots+k^7)=\frac{1}{8}k^4(k+1)^4 \ldots(2)$

For $P(k+1)$:

(a) $1+2+\cdots+k+(k+1)=\frac{k(k+1)}{2}+(k+1)=\frac{(k+1)}{2}[k+2]$, by (1)

(b) $[1^5+2^5+\cdots+k^5+(k+1)^5]+[1^7+2^7+\cdots+k^7+(k+1)^7]$

$=\frac{1}{8}k^4(k+1)^4+(k+1)^5+(k+1)^7$, by (2)

$=\frac{1}{8}(k+1)^4[k^4+8(k+1)+8(k+1)^3]$

$=\frac{1}{8}(k+1)^4[k^4+8k+8+8(k^3+3k^2+3k+1)]$

$=\frac{1}{8}(k+1)^4[k^4+8k^3+24k^2+32k+16]$

$=\frac{1}{8}(k+1)^4(k+2)^4$

$\therefore P(k+1)$ is true.

By the Principle of mathematical induction, $P(n)$ is true for all $n \in \boldsymbol{N}$.

6. Prove $5^{2n+1}+11^{2n+1}+17^{2n+1}$ is divisible by 33 for all non-negative integer value of n.

Let $P(n): 5^{2n+1}+11^{2n+1}+17^{2n+1}=33a_n$ where $a_n \in \boldsymbol{N}, n \in \boldsymbol{N} \cup \{0\}$.

For $P(0): 5^{2(0)+1}+11^{2(0)+1}+17^{2(0)+1}=33=33a_0,\ a_0=1$

Assume $P(k)$ is true for some $k \in N$

We have $5^{2k+1}+11^{2k+1}+17^{2k+1}=33a_k \ldots(1)$

For $P(k+1)$,

$5^{2k+3}+11^{2k+3}+17^{2k+3}$

$=25(5^{2k+1})+121(11^{2k+1})+289(17^{2k+1})$

$=25[5^{2k+1}+11^{2k+1}+17^{2k+1}]+96(11^{2k+1})+264(17^{2k+1})$

$=25[33a_k]+(96)(11)(11^{2k})+264(17^{2k+1})$, by (1)

$=33[25a_k+32(11^{2k})+8(17^{2k+1})]$

$=33a_{k+1}$

$\therefore P(k+1)$ is true.

By the Principle of mathematical induction, $P(n)$ is true for all $n \in \boldsymbol{N} \cup \{0\}$.

7. Prove, by Mathematical Induction, that $n(n+1)(n+2)(n+3)\dots(n+r-1)$ is divisible by $r!$ for all natural numbers n, where $r = 1,2,\dots$

Let $P(n)$: $n(n+1)(n+2)(n+3)\dots(n+r-1) = r!$ is divisible by $r!$

That is, n is divisible by $1!$

$n(n+1)$ is divisible by $2!$

$n(n+1)(n+2)$ is divisible by $3!$

$n(n+1)(n+2)(n+3)$ is divisible by $3!$

…

$n(n+1)(n+2)(n+3)\dots(n+r-1)$ is divisible by $r!$

For $P(1)$, $1(1+1)(1+2)(1+3)\dots(1+r-1) = r!$ is obviously divisible by $r!$.

Assume $P(k)$ is true for some $k \in \boldsymbol{N}$, that is

That is, k is divisible by $1! \dots eq(1)$

$k(k+1)$ is divisible by $2! \dots eq(2)$

$k(k+1)(k+2)$ is divisible by $3! \dots eq(3)$

$k(k+1)(k+2)(k+3)$ is divisible by $4! \dots eq(4)$

…

$k(k+1)(k+2)(k+3)\dots(k+r-2)$ is divisible by $(r-1)! \dots eq(r-1)$

$k(k+1)(k+2)(k+3)\dots(k+r-1)$ is divisible by $r! \dots eq(r)$

For $P(k+1)$,

(1) $k+1$ is divisible by $1!$ is true

(2) $(k+1)(k+2) = k(k+1) + 2(k+2)$ is divisible by $2!$ since

by $eq(2)$ $k(k+1)$ is divisible by $2!$ and $2(k+2)$ is also divisible by $2!$

(3) $(k+1)(k+2)(k+3) = k(k+1)(k+2) + 3(k+1)(k+2)$

$k(k+1)(k+2)$ is divisible by $3!$, by $eq(3)$

$(k+1)(k+2)$ is divisible by $2!$ (just proved in (2))

$3(k+1)(k+2)$ is divisible by $3 \times 2! = 3!$

Therefore $(k+1)(k+2)(k+3)$ is divisible by $3!$

(4) $(k+1)(k+2)(k+3)(k+4) = k(k+1)(k+2)(k+3) + 4(k+1)(k+2)(k+3)$

$k(k+1)(k+2)(k+3)$ is divisible by $4!$ by $eq(4)$

$(k+1)(k+2)(k+3)$ is divisible by $3!$ (just proved in (3))

$4(k+1)(k+2)(k+3)$ is divisible by $4 \times 3! = 4!$

Therefore $(k+1)(k+2)(k+3)(k+4)$ is divisible by $4!$

(5) …. Up to $(r-1)$

Continue in this way, by writing

$(k+1)(k+2)(k+3)\dots(k+r-1)(k+r)$

$= k(k+1)(k+2)(k+3)\dots(k+r-1) + r(k+1)(k+2)(k+3)\dots(k+r-1)$

is divisible by $r!$ since

$k(k+1)(k+2)(k+3)\dots(k+r-1)$ is divisible by $r!$ by $eq(r)$

$(k+1)(k+2)(k+3)\dots(k+r-1)$ is divisible by $(r-1)!$

$r(k+1)(k+2)(k+3)\ldots(k+r-1)$ is divisible by $r(r-1)! = r!$

Therefore $(k+1)(k+2)(k+3)\ldots(k+r-1)(k+r)$ is divisible by $r!$

$\therefore P(k+1)$ is true.

By the Principle of mathematical induction, $P(n)$ is true $\forall n \in \boldsymbol{N}$.

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